# Modern Cryptography: Lecture 15

*Selected Topics*

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# Organizational

- Where to find the slides and homework?
  - https://danielslamanig.info/ModernCrypto18.html
- How to contact me?
  - daniel.slamanig@ait.ac.at
- Tutor: Karen Klein
  - karen.klein@ist.ac.at

- Official page at TU, Location etc.
  - https://tiss.tuwien.ac.at/course/courseDetails.xhtml?dswid=8632&dsrid=679&courseNr=192062&semester=2018W
- Tutorial, TU site
  - https://tiss.tuwien.ac.at/course/courseAnnouncement.xhtml?dswid=5209&dsrid=341&courseNumber=192063&courseSemester=2018W

- Exam for the second part: Thursday 31.01.2019 15:00-17:00 (Tutorial slot)

# Topics in Advanced Cryptography...

Etc.

Etc.

Oblivious RAM, Private Information Retrieval

Privacy Enhancing Cryptography (Crypto PETs)

Fully Homomorphic Encryption (Computing on Encypted Data)

Program Obfuscation

Etc.

Functional Encryption

IBE

ABE

(Non-Interactive) Zero Knowledge Proofs (+ SNARKs, STARKs, etc.)

Verifiable Computation

Multiparty Comptuation (MPC)

Threshold Cryptography

Public Key Encryption

Digital Signatures

Key Exchange

Oblivious Transfer

Etc.

Hash functions

Symmetric Encryption

MACs

PRFs

Commitments

# Selected Topics

- Threshold cryptography
  - Distribute operations with the secret key sk among a set of parties
  - A certain number of participants need to be involved to perform an operation
- Brief primer on Multiparty Computation (MPC)
  - A number of parties can compute any function jointly without revealing their inputs to the other parties
- Puncturable Encryption
  - Public key encryption with “update capabilities” on the secret key
  - Secret key can be punctured on ciphertext s.t. this ciphertext can no longer be decrypted

# Threshold Cryptography: Motivation

- If the secret key (of an encryption or signature scheme) is in a single location, this represents a single point of failure
  - Problem that happened in practice, e.g., with Bitcoin ECDSA private keys
- We may want to enforce that a signature generation or decryption is only possible when a certain set of participants agree to do so

- Idea
  - Let a set of parties jointly generate a secret key ("shares" of the key may also be distributed to the parties by a trusted dealer)
  - The public key typically looks like a public key of the underlying scheme
    - So public key operations are as usual
  - Using the secret key (i.e., signing or decryption) requires an interactive protocol between (a subset of the) participants

# Secret Sharing

- A dealer shares a secret key between n participants
- Each participant $i \in \{1, \ldots, n\}$ receives a share
- Predefined groups of participants (so called authorized groups) can cooperate to reconstruct the secret from their shares
- Unauthorized groups cannot get any information about the secret

- We will look at (k, n)-threshold secret sharing schemes
  - Every subset of at least k participants of the n participants can reconstruct the secret (is authorized)
  - Any subset of k–1 participants can get no any information about the secret (is unauthorized)

# (n,n)-Treshold Secret Sharing

- Let s be a secret from a finite group (G, +)
- The dealer chooses n-1 uniformly random elements $s_1, \ldots, s_{n-1}$ from G and computes $s_n = s - (s_1 + \ldots + s_{n-1})$
- The shares are $(s_1, \ldots, s_n)$ and party i is given share $s_i$
- Given $(s_1, \ldots, s_n)$, one can successfully recover $s = s_1 + \ldots + s_n$
- Given $s_i$ for $i \neq j$: $\Sigma_{i \neq j} s_i = s - s_j$ is uniformly random (no information)

- Not robust at all!
  - If a single participant fails to provide the share reconstruction is not possible
- We are interseted in (k,n)-threshold schemes where k<n

# Shamir Secret Sharing

- Basis
  - Given k points on the plane $(x_1, y_1), \ldots, (x_k, y_k)$, all $x_i$ distinct, there exists an unique polynomial f of degree $\leq k - 1$, s.t. $f(x_i) = y_i$ for all i
  - Holds also in the field $\mathbb{Z}_p$ for p prime
  - Constructive proof: Use Lagrange interpolation
- How is this used?
  - Let s be a secret in $\mathbb{Z}_p$
  - Dealer selects a random degree k-1 polynomial $f(x) = a_{k-1}x^{k-1} + \ldots + a_1x + a_0$
    - Select $a_{k-1}, \ldots, a_1$ uniformly random from $\mathbb{Z}_p$ and set $a_0 = s$
  - For $i \in \{1, \ldots, n\}$, give the share $s_i = (i, f(i))$ to the $i^{th}$ participant

# Shamir Secret Sharing

- **Correctness:** the secret s can be reconstucted from every subset of k shares
  - Proof: By the Langrange formula, given k points $(x_i, y_i)$, for i = 1, … ,k

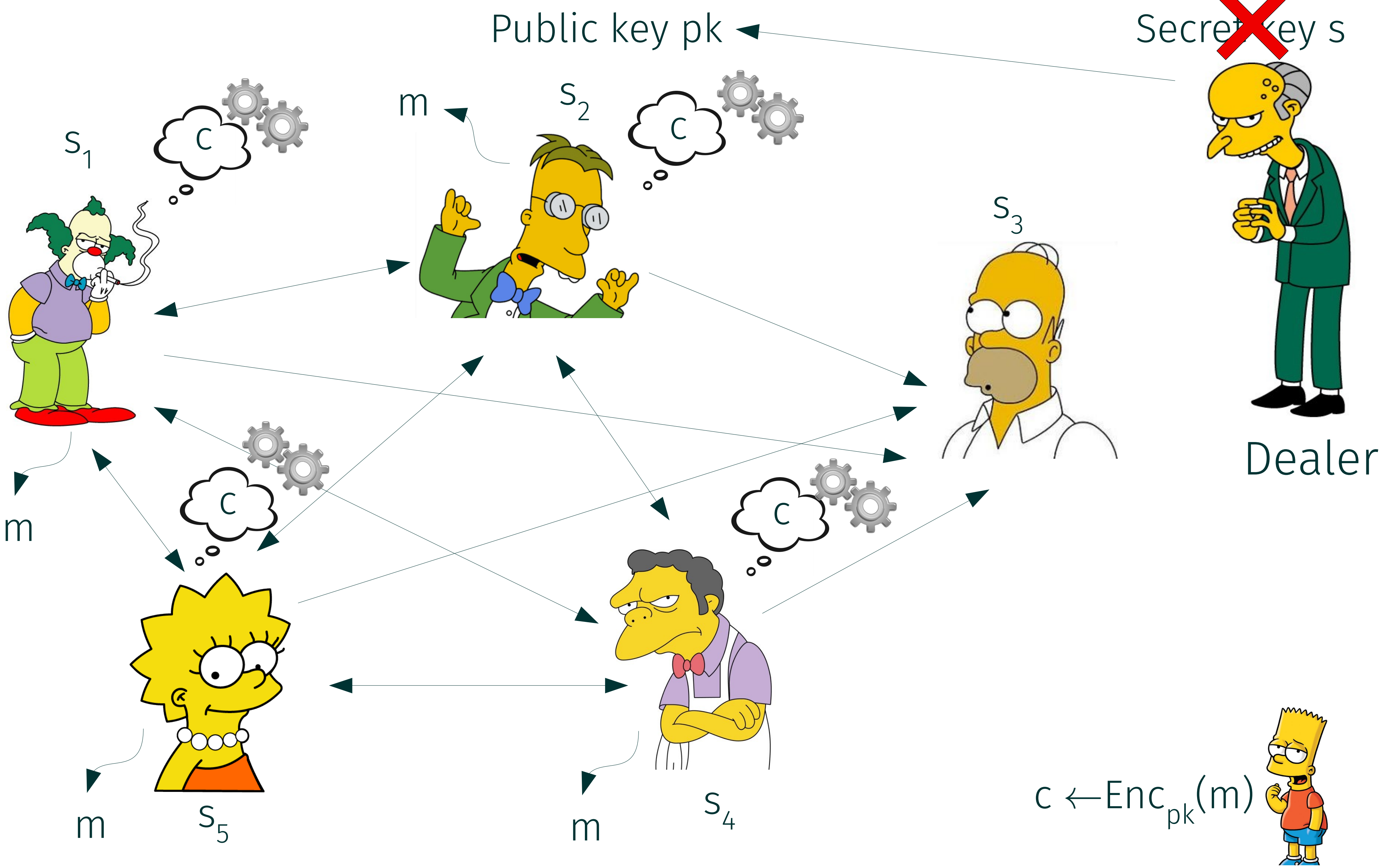
$$f(x) = \sum_{i=1}^{k} y_i \prod_{j=1, j\neq i}^{k} \frac{x - x_j}{x_i - x_j} \bmod p$$

  - and consequently

$$s := f(0) = \sum_{i=1}^{k} y_i \underbrace{\prod_{j=1, j\neq i}^{k} \frac{-x_j}{x_i - x_j} \bmod p}_{\Delta_i}.$$

- **Secrecy (perfect):** Any subset of up to k – 1 shares does not leak any information on the secret
  - Proof: Given k – 1 shares $(x_i, y_i)$ every candidate secret $s' \in \mathbb{Z}_p$ corresponds to a unique polynomial of degree k-1 for which f(0)=s'. For all $s' \in \mathbb{Z}_p$ the probabilities Pr[s' = s] are equal.

# Threshold Encryption



# Threshold ElGamal Encryption

- Let $x$ be the ElGamal secret key and $y := g^x$ the public key (we work in a group G of prime order q generated by g).
- Every participant i receives a share $s_i = (i, x_i)$ of $x$ obtained from Shamir (k,n)-threshold secret sharing
- We observe (notice that $\Delta_j$ are publicly computable) that

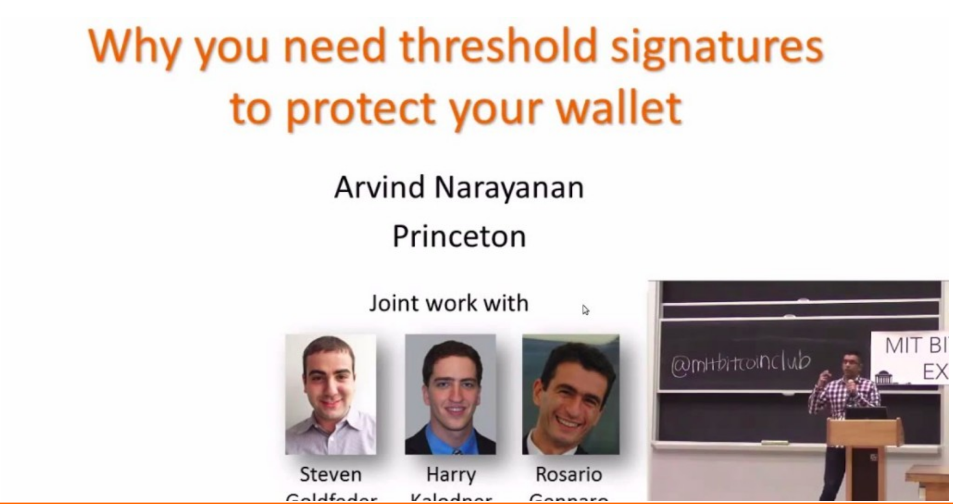
$$x = \sum_{j \in X} x_j \Delta_j \text{ and } g^x = \prod_{j \in X} (g^{x_j})^{\Delta_j}$$

- Given an ElGamal ciphertext $(c_1, c_2) = (g^r, my^r)$ we assume a honst set X of t participants
  - Every participant j in X broadcasts $w_j := (c_1)^{x_j}$
  - Everyone in X can recover the plaintext as $m = \dfrac{c_2}{\prod_{j \in X} w_j^{\Delta_j}}$

Correctness: $\dfrac{c_2}{\prod_{j \in X} w_j^{\Delta_j}} = \dfrac{m(g^x)^r}{g^{r \sum_{j \in X} x_j \Delta_j}} = \dfrac{mg^{rx}}{g^{rx}}$

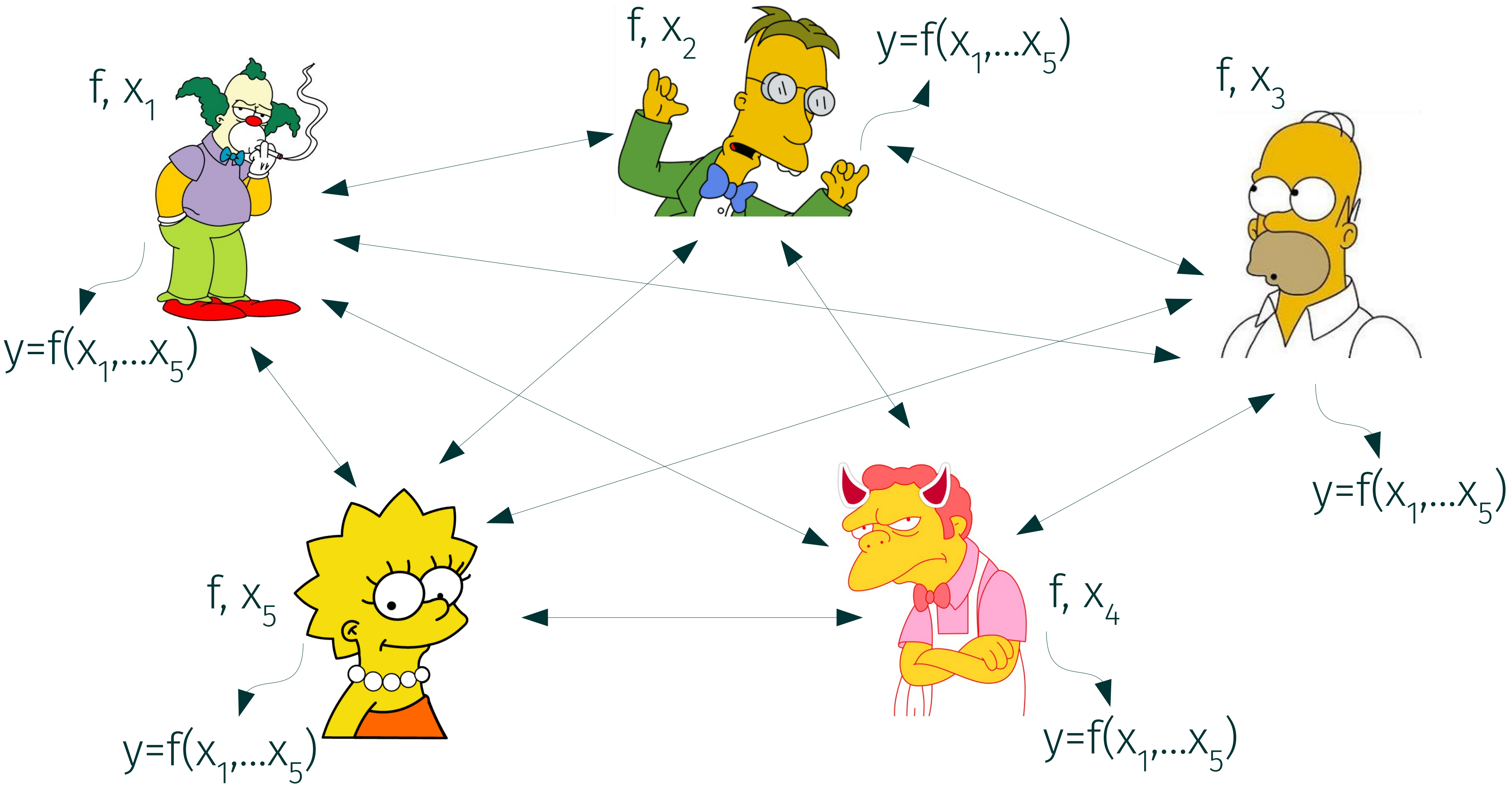
# Threshold Cryptography: Remarks

- We have assumed that the parties participating in the decryption are honest
  - Malicious parties can enforce an incorrect result by publishing a malformed $w_j$ value
  - Can be prevented by forcing the parties to prove that the $w_j$ values are well formed (i.e., by attaching a non-interactive zero-knowledge proof)
- Can come up with threshold versions of various signature schemes
  - Schnorr, (EC)DSA, etc.
  - Somewhat hot topic today (cryptocurrencies)




# Multiparty Computation

- We have seen a very specific functionality computed in a distributed way without requiring the participants to reveal their secret inputs
- Can we do every computation in such a threshold manner? Yes!



# Multiparty Computation

- We look at Ben-Or, Goldwasser, Wigderson (BGW) in a finite field $\mathbb{Z}_p$
  - Every possible function in $\mathbb{Z}_p$ is a polynomial
  - We need to show how we can do addition and multiplication
- BGW is a general MPC protocol that provides information theoretic guarantees
  - in the presence of semi-honest adversaries controlling a minority of parties ( $< n/2$)
  - in the presence of malicious adversaries controlling less than a third of the parties ($< n/3$).

# Multiparty Computation

- Use Shamir's (k,n)-threshold secret sharing with $k > n/2$ (honest majority)
- Every party i has a secret $s_i$ and polynomial $f_i(0) = s_i$
- Every party j holds shares $f_i(j)$, $i \neq j$,

- **Addition:** Given $f_1(j)$ and $f_2(j)$ just add the shares: participants then share the polynomial $f_1 + f_2$ with $(f_1 + f_2)(0) = s_1 + s_2$.
- **Multiplication:** if $h = (f_1 \cdot f_2)$ then $h(0) = s_1 \cdot s_2$
  - However, h would have degree $\deg f_1 + \deg f_2 = 2k - 2$
  - Coefficients of h are not uniformly random
  - After every multiplication the parties perform a simple protocol that reduces the degree of h and adds uniformly random values to all coefficients of h, except to $h_0$

# Puncturable Encryption

- Public key encryption with “update capabilities” on the secret key
- Secret key can be punctured on ciphertext s.t. this ciphertext can no longer be decrypted

Conventional encryption scheme:

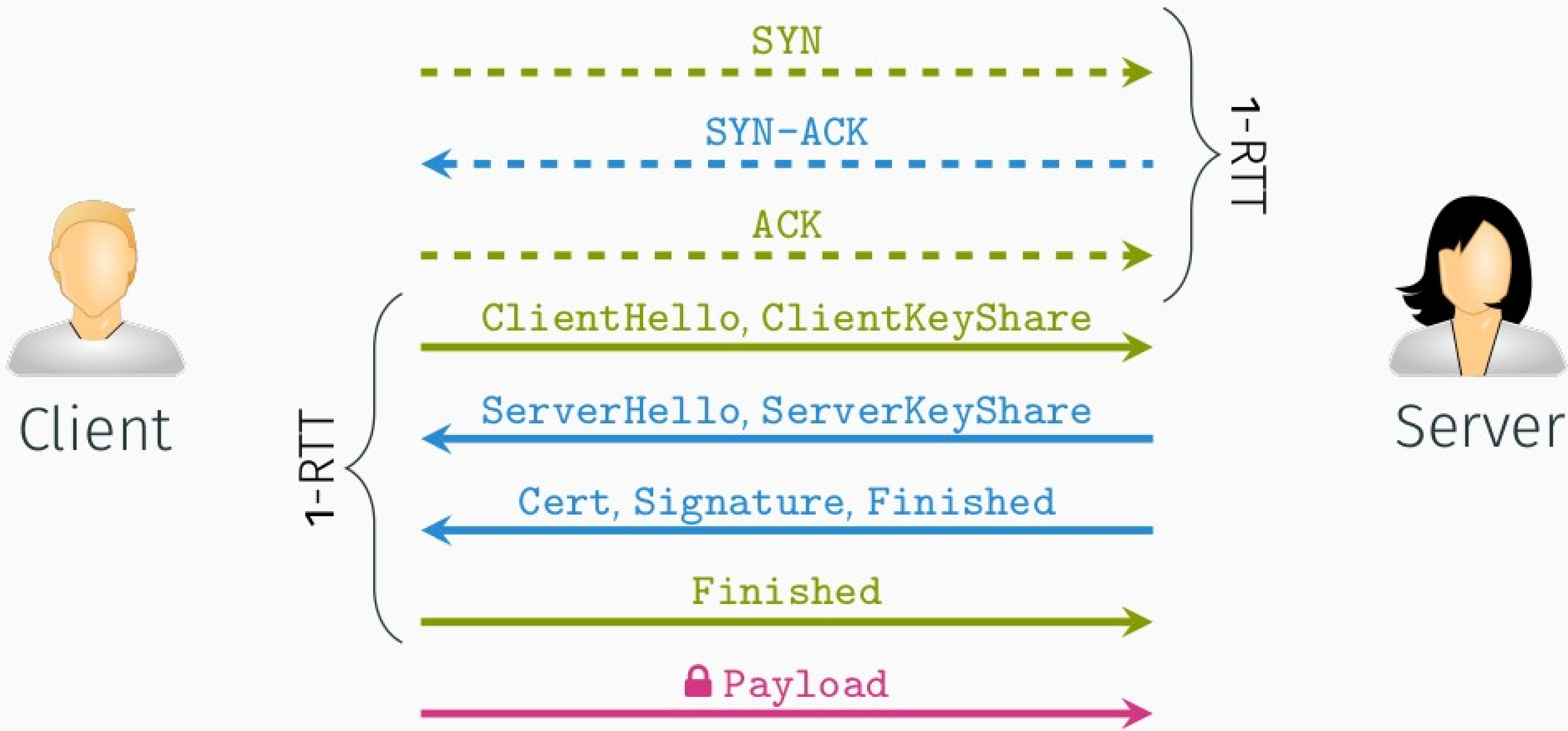
- (KeyGen, Enc, Dec)

\+ Additional algorithm 🔑′ ← Punc(🔑, $C$)

Properties

- 🔑′ no longer useful to decrypt $C$
- 🔑′ still useful to decrypt other ciphertexts
- Repeated puncturing possible

# Puncturable Encryption: Application fs 0-RTT Key Exchange

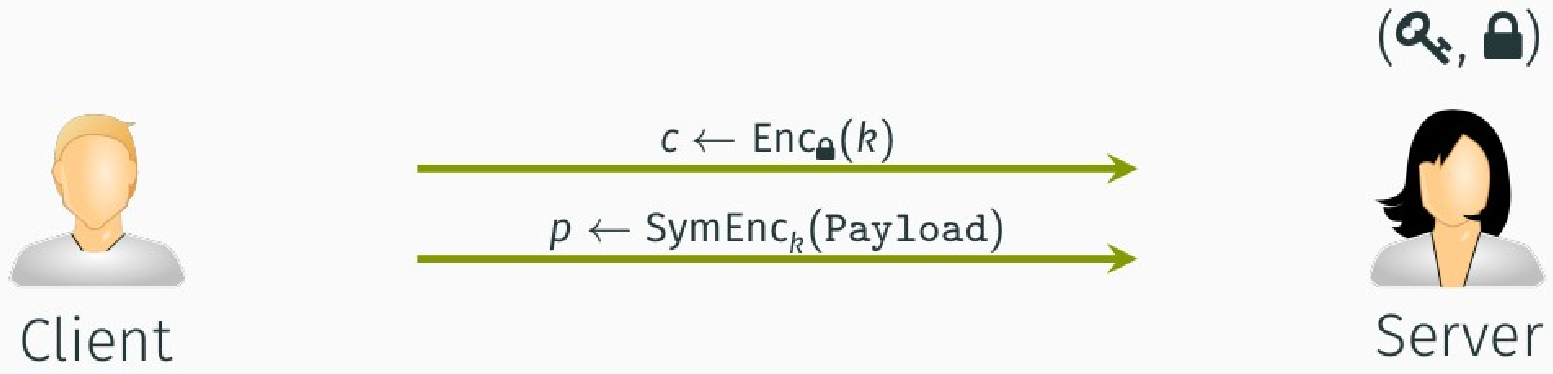



Can we already send encrypted payload with the first message in the second round?

Desired properties:

- Replay protection
- Forward secrecy

# Puncturable Encryption: Application fs 0-RTT Key Exchange




Major deficiencies:

- No forward secrecy
- Vulnerable to replay attacks

# Puncturable Encryption

- We are looking at one construction idea
  - Construct a scheme with non-negligible correctness error: does not matter too much for key-exchange
  - E.g., 1 in 1000 sessions fail (can then fallback to 1-RTT)

- The most basic construction is called Bloom Filter Encryption (BFE)
  - Bloom Filter: data structure for probabilistic set membership checks

# Puncturable Encryption: Bloom Filters

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | | | | | | | | | | | | | | | | | | m |

- Initial state $T := 0^m$
- $k$ universal hash functions $(H_j)_{j \in [k]}$
- $H_j : \mathcal{U} \to [m]$
- Throughout this talk, let $k = 3$

# Puncturable Encryption: Bloom Filters

$\{x, y, z\}$

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | | | | | | | | | | | | | | | | | | m |

- Initial state $T := 0^m$
- $k$ universal hash functions $(H_j)_{j \in [k]}$
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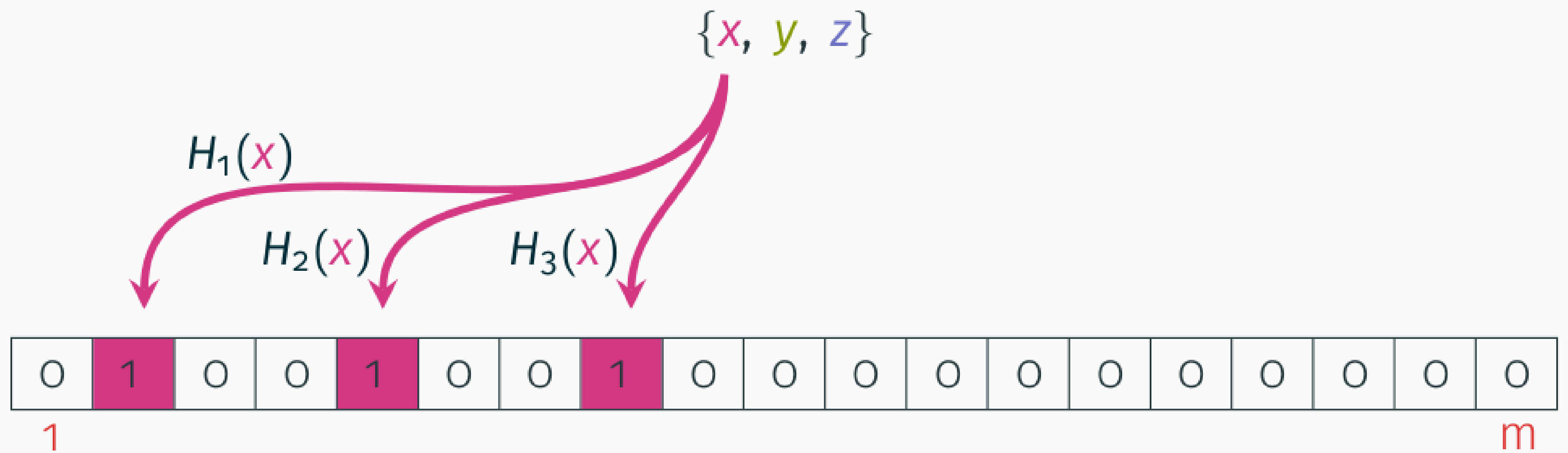
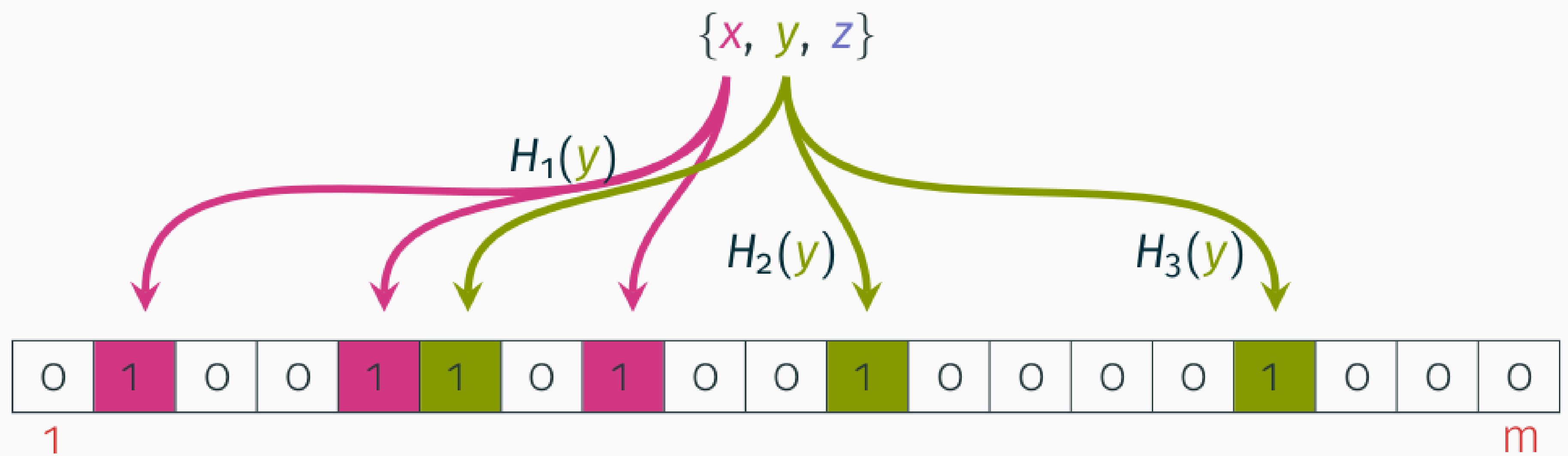
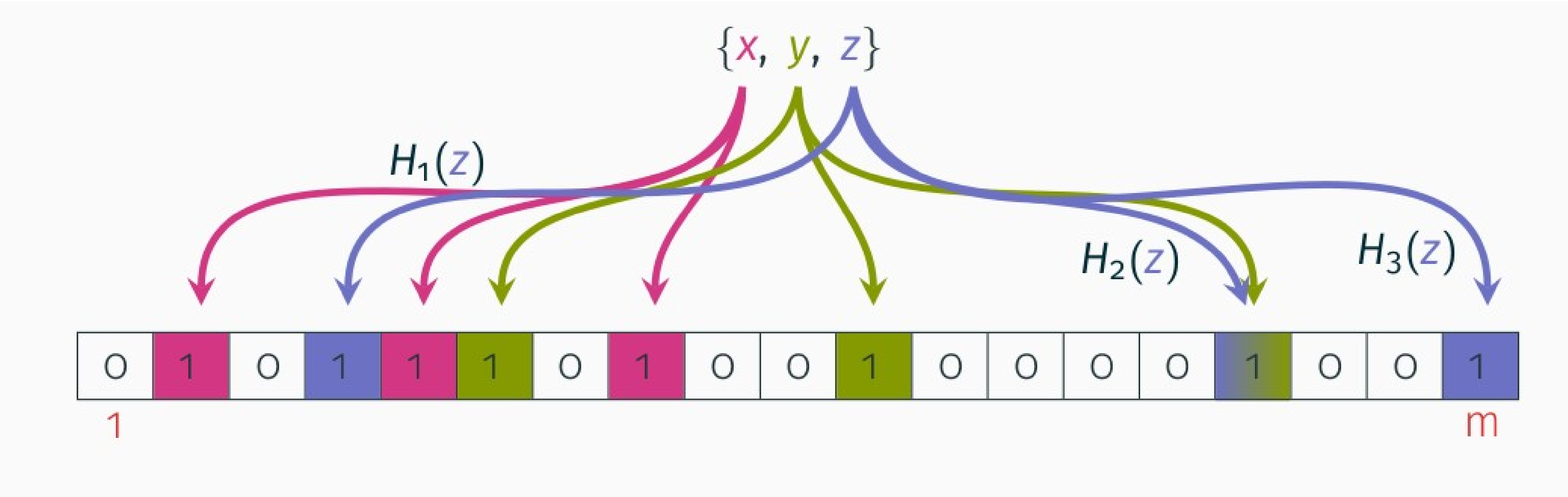
# Puncturable Encryption: Bloom Filters



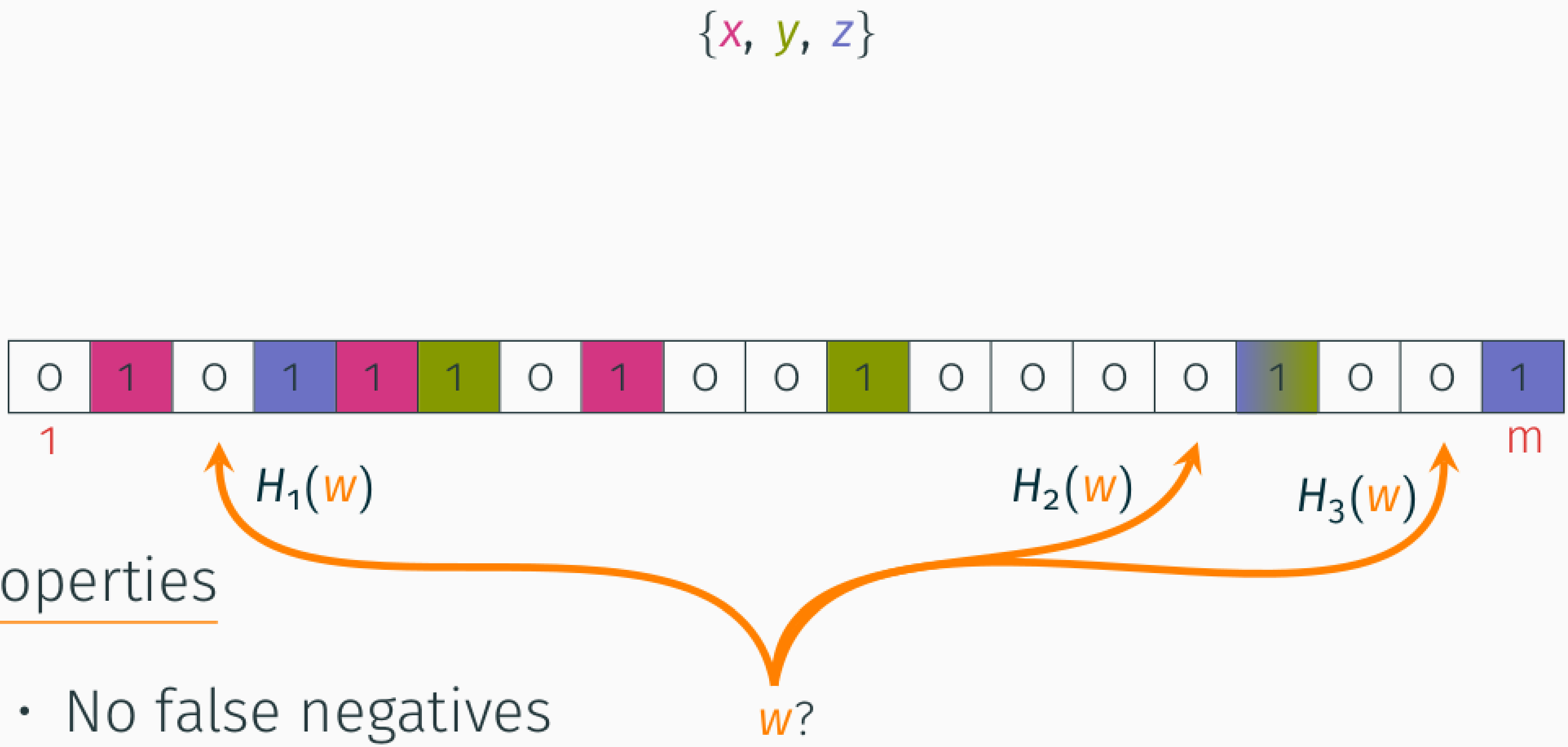
# Puncturable Encryption: Bloom Filters



# Puncturable Encryption: Bloom Filters



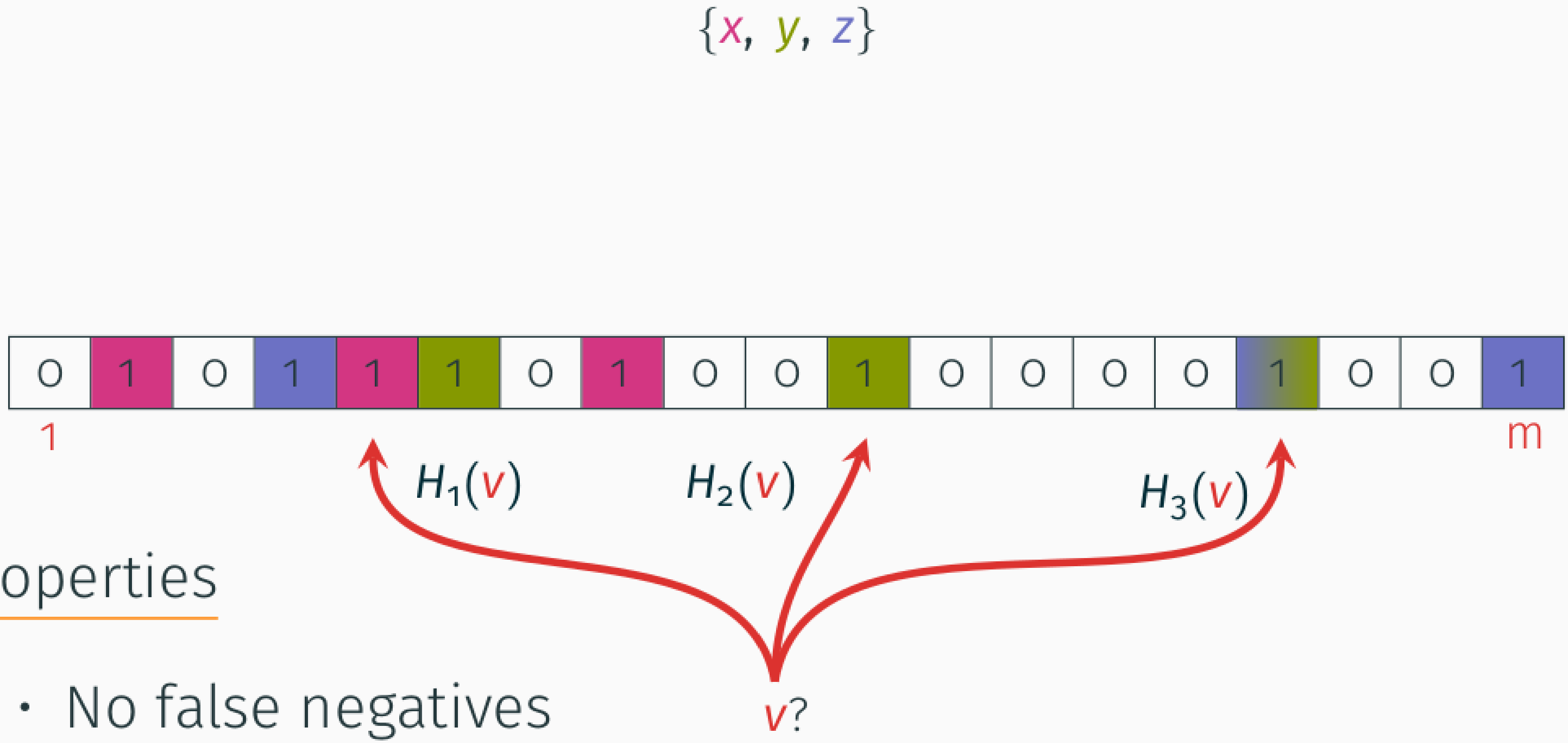
# Puncturable Encryption: Bloom Filters




## Properties

- No false negatives

# Puncturable Encryption: Bloom Filters

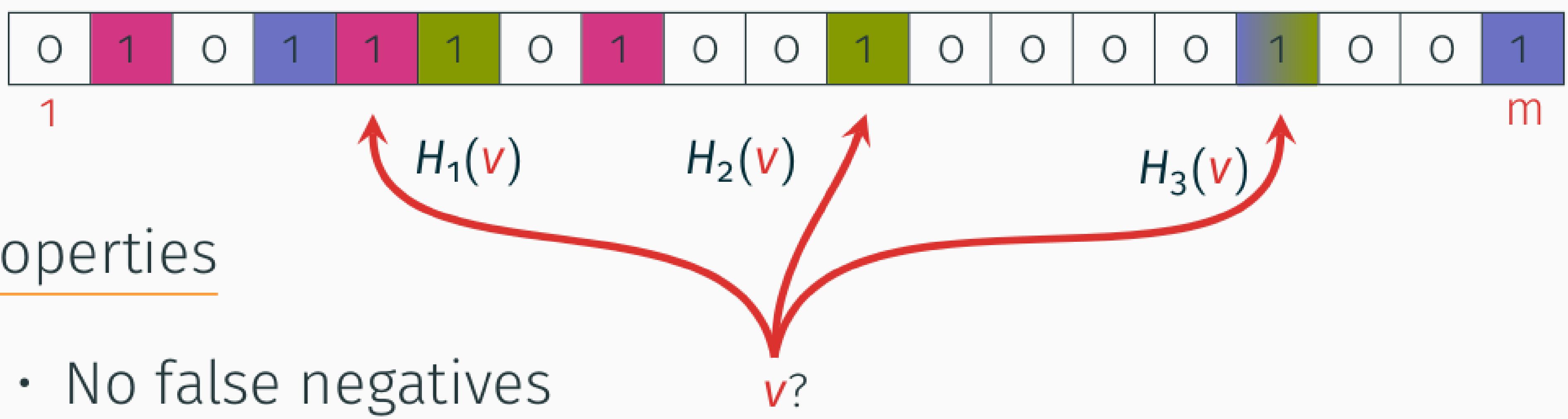



## Properties

- No false negatives
- False positives possible

# Puncturable Encryption: Bloom Filters

$\{x, y, z\}$




## Properties

- No false negatives
- **False positives possible**
- Probability determined by $k$, $m$, and # inserted elements

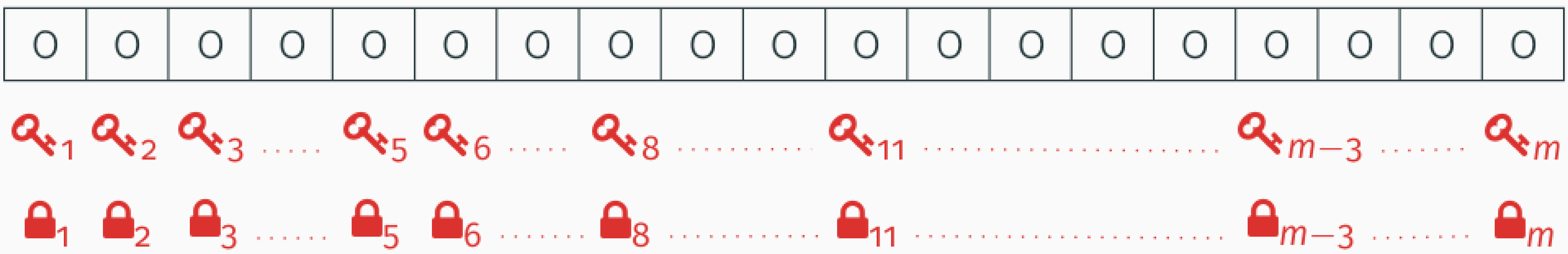
# Puncturable Encryption: BFE Construction

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

KeyGen

- Set up BF

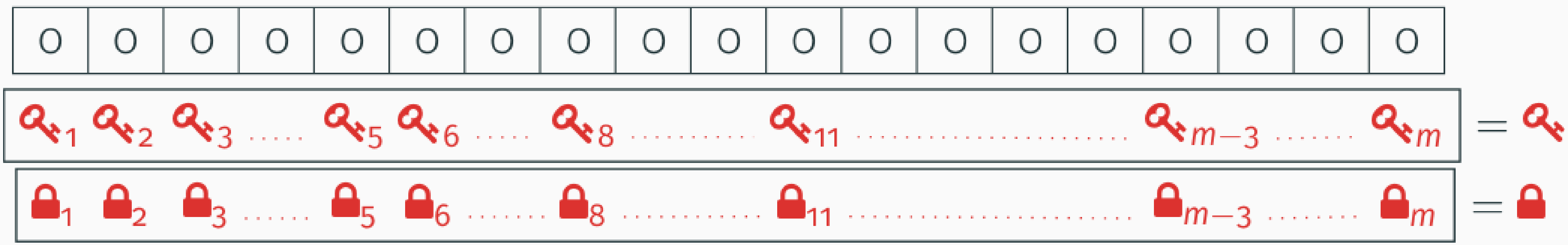
# Puncturable Encryption: BFE Construction




KeyGen

- Set up BF
- Associate key pair to each bit

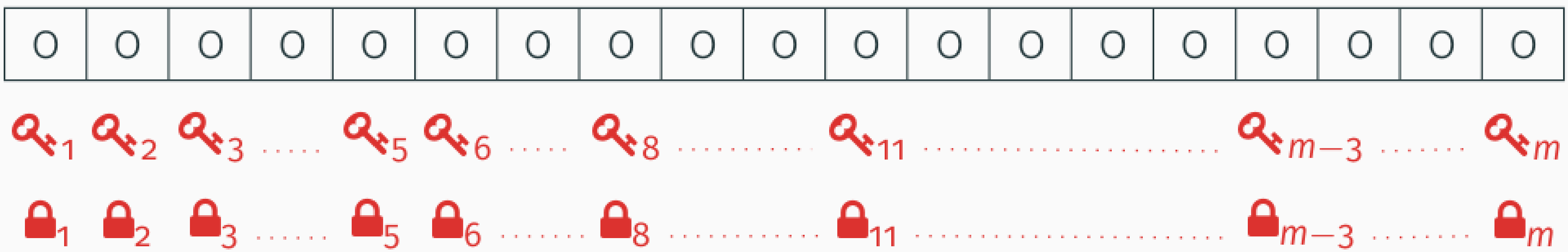
# Puncturable Encryption: BFE Construction




KeyGen

- Set up BF
- Associate key pair to each bit
- Compose BFE key pair ( , )

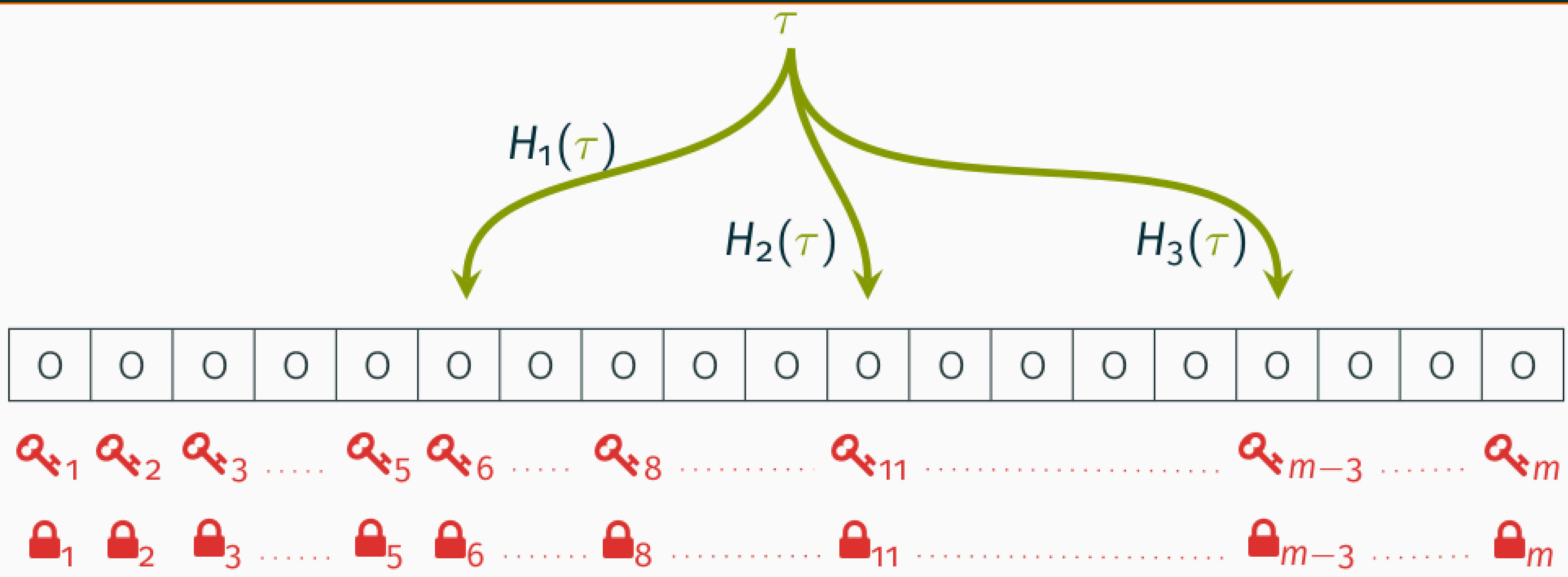
# Puncturable Encryption: BFE Construction




## Encrypt message $M$

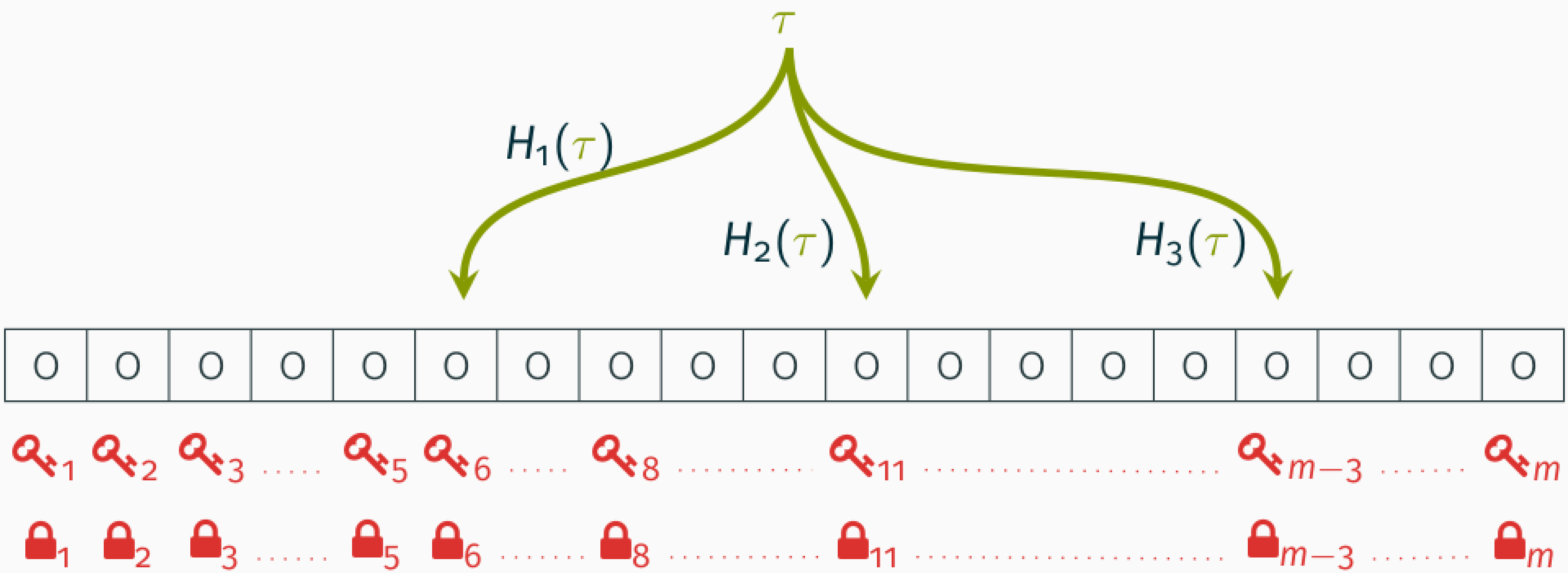
- Randomly choose tag $\tau$

# Puncturable Encryption: BFE Construction




**Encrypt message $M$**

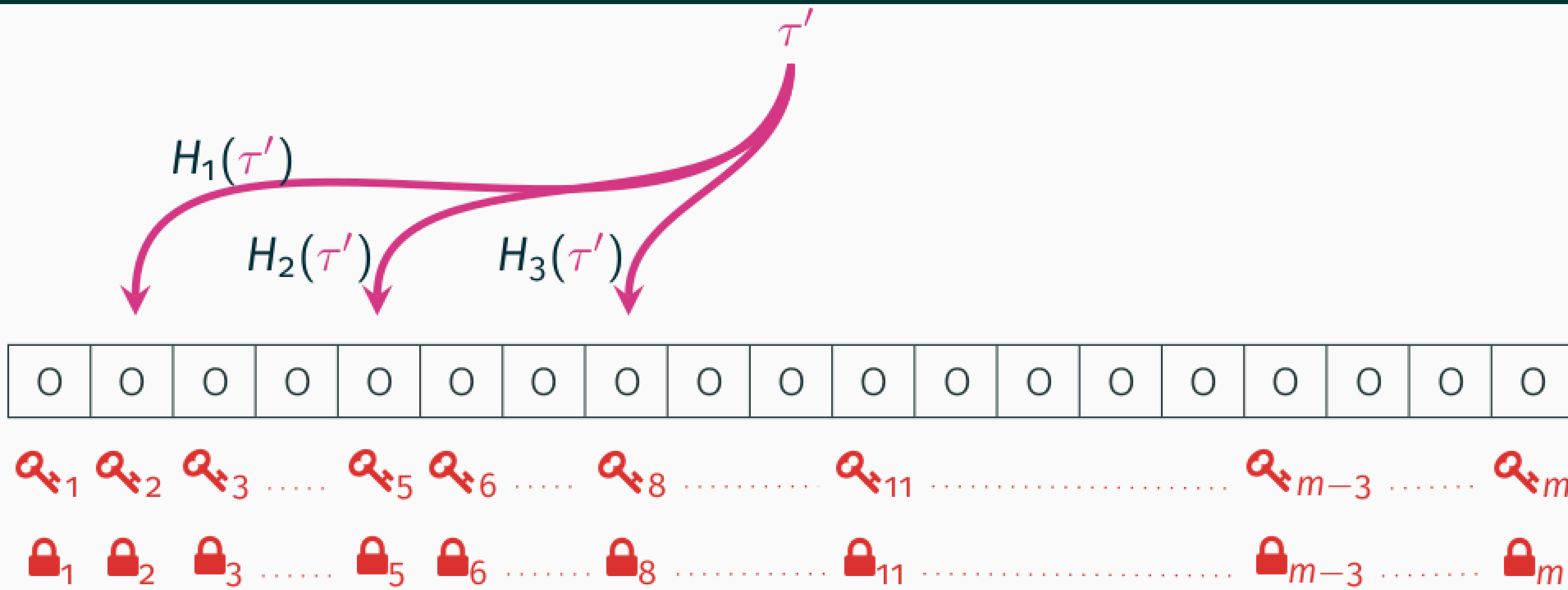
- Randomly choose tag $\tau$
- Determine indexes from $\tau$




## Encrypt message $M$

- Randomly choose tag $\tau$
- Determine indexes from $\tau$
- $C_\tau \leftarrow \mathrm{Enc}_{6 \vee 11 \vee m-3}(M)$

# Puncturable Encryption: Bloom Filters

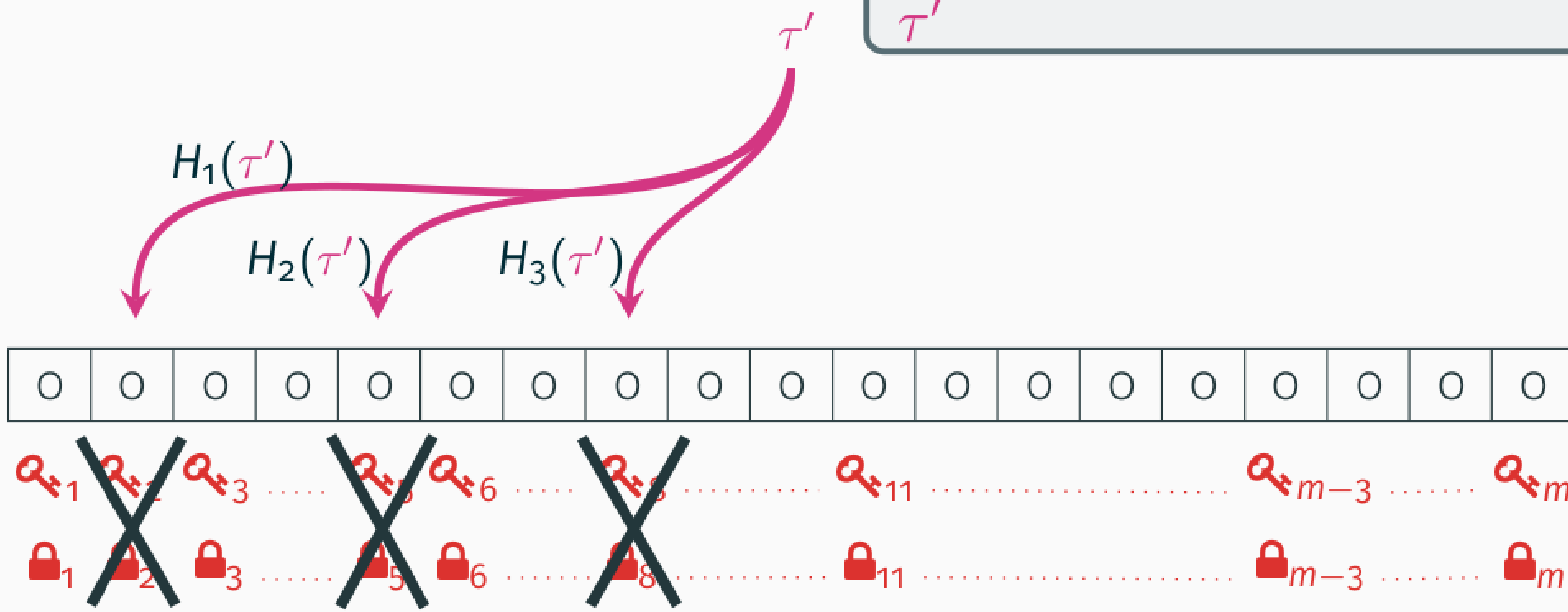



**Puncture ciphertext $C_{\tau'}$**

- Determine BF indexes from $\tau'$

# Puncturable Encryption: Bloom Filters

Secret key no longer useful to decrypt $C_{\tau'}$ with associated tag $\tau'$

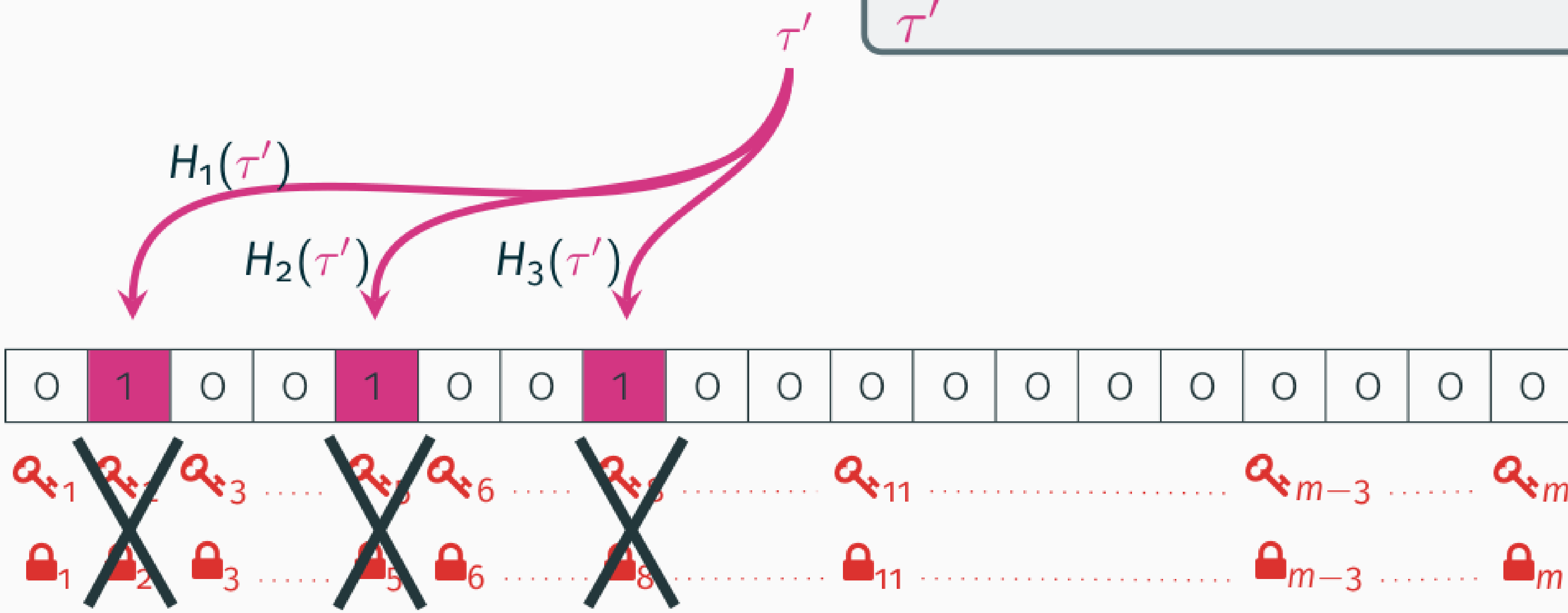



**Puncture ciphertext $C_{\tau'}$**

- Determine BF indexes from $\tau'$
- Delete associated keys

# Puncturable Encryption: Bloom Filters

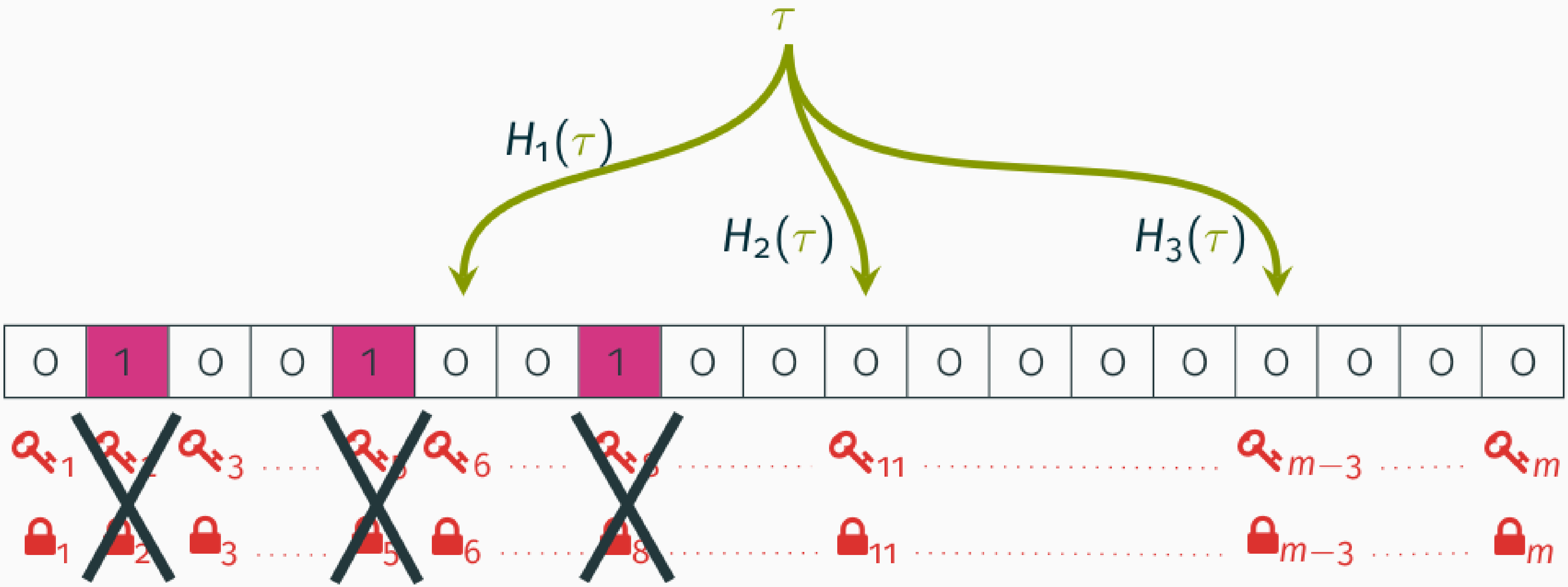
Secret key no longer useful to decrypt $C_{\tau'}$ with associated tag $\tau'$




**Puncture ciphertext $C_{\tau'}$**

- Determine BF indexes from $\tau'$
- Delete associated keys
- Update BF state

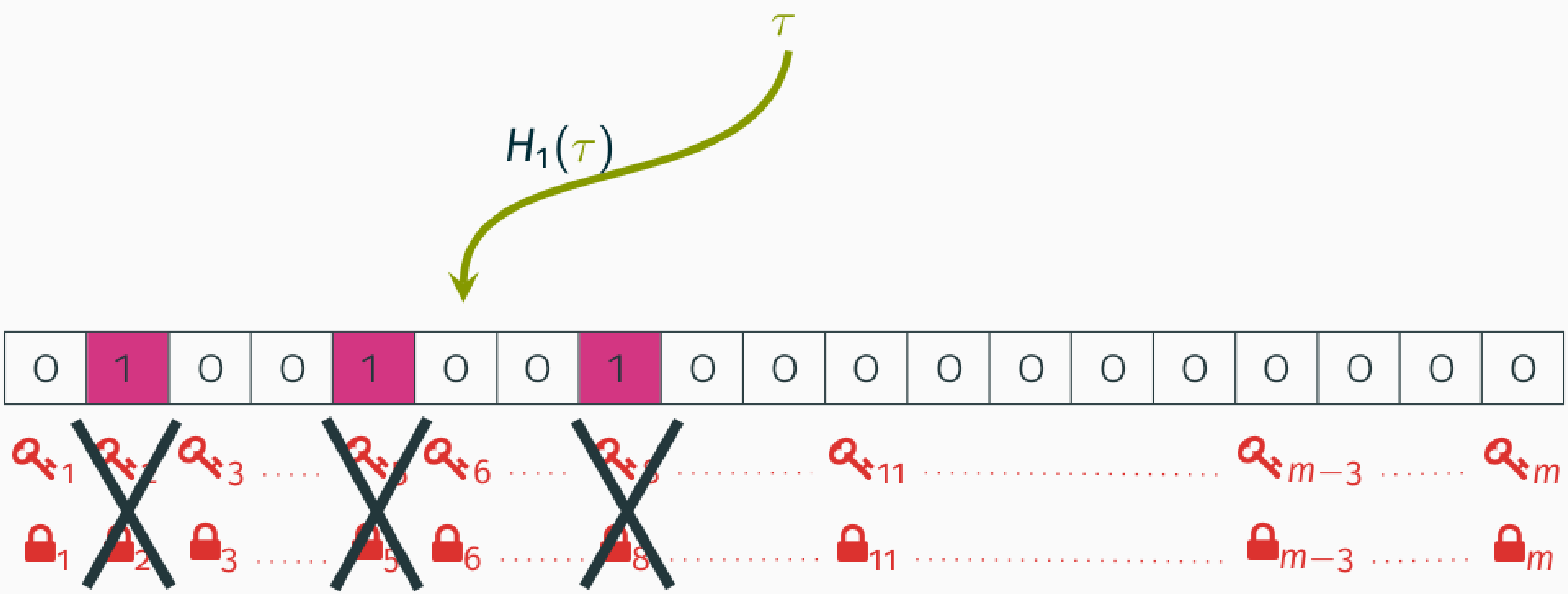
# Puncturable Encryption: Bloom Filters




## Decrypt ciphertext $C_\tau$

- Determine BF indexes from $\tau$

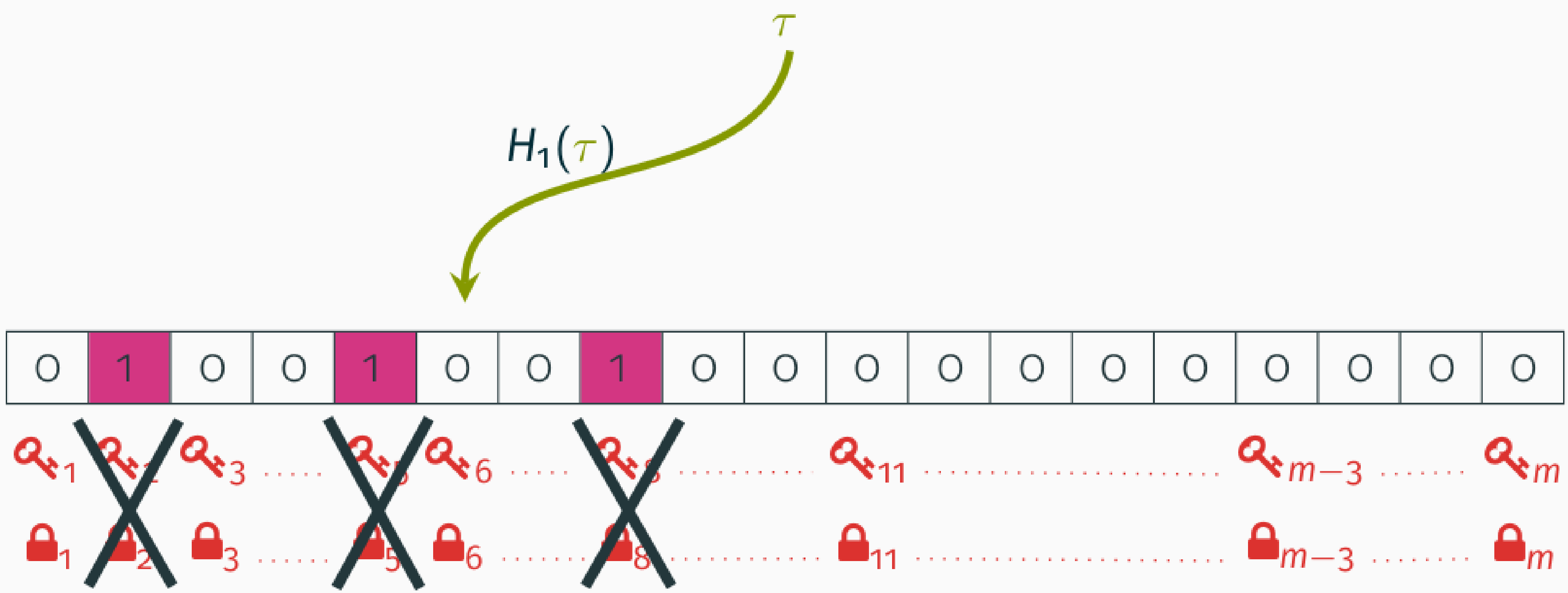
# Puncturable Encryption: Bloom Filters




## Decrypt ciphertext $C_\tau$

- Determine BF indexes from $\tau$
- Let $i$ lowest index w. $BF[i] = 0$

# Puncturable Encryption: Bloom Filters




## Decrypt ciphertext $C_\tau$

- Determine BF indexes from $\tau$
- Let $i$ lowest index w. $BF[i] = 0$
- $M \leftarrow \mathsf{Dec}_{6}(C_\tau)$

# Puncturable Encryption: Bloom Filters

- Maximum # of elements in BF: $2^{20}$
  - $\approx 2^{12}$ puncturings/day for full year
- False positive probability: $10^{-3}$
- BF size $m = n \cdot \ln p/(\ln 2)^2 \approx 2\text{MB}$
- # hash functions $k = \lceil m/n \cdot \ln 2 \rceil = 10$

- Constructions from different primitives
  - Identity-based encryption (IBE), Attribute-based encryption (ABE)
  - Identity-based broadcast encryption (IBBE)

| Construction | \|🔒\| | \|🔑\| | \|C\| | Dec | Punc |
|---|---|---|---|---|---|
| IBE [Crypto'01] | $O(1)$ | $O(m)$ | $O(k)$ | $O(k)$ | $O(k)$ |
| ABE [CT-RSA'13, AC'15] | $O(m)$ | $O(m^2)$ | $O(1)$ | $O(k)$ | $O(k)$ |
| IBBE [AC'07] [1] | $O(k)$ | $O(m)$ | $O(1)$ | $O(k)$ | $O(k)$ |

# The End

- Thank you all for participating in the course! It was a lot of fun!
- If you are interested in summer internships/bachelor/master projects please just contact me

Good luck for the final exam!!